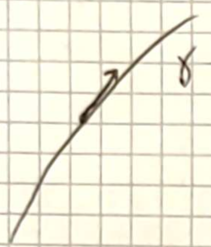
Riemannian Geometry $\boxed{\text{Week 6}}$

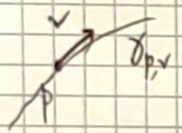
Last week: We defined the notion of a geodesic



$$\nabla_{\dot\gamma}\dot\gamma = 0$$

in local coordinates: $\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0$

From ODE theory: $\forall p \in M$, $\forall v \in T_pM$: $\exists!$ <u>maximal</u> geodesic



$$\gamma_{p,v} : (-T_-, T_+) \to M, \qquad \gamma_{p,v}(0) = p, \quad \dot\gamma_{p,v}(0) = v.$$

In the case of geodesics of a Riemannian manifold, what is the relation between geodesics and distances?

As we will see:

- A length minimizing curve is <u>always</u> a geodesic

but

- A geodesic between two points in <u>not always</u> length minimizing.

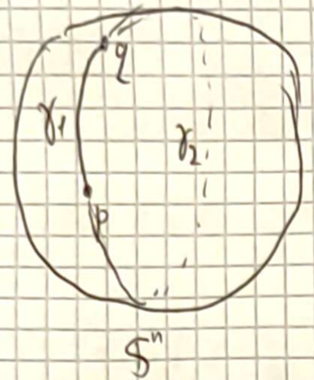
<u>Example</u>: On $(\mathbb{S}^n, g_{\mathbb{S}^n})$ $\quad$ ($g_{\mathbb{S}^n}$: induced metric from $(\mathbb{R}^{n+1}, g_E)$)

as we have seen $\nabla_X Y = \pi^\perp(D_X Y)$

D → Flat connection on $\mathbb{R}^{n+1}$

$\pi^\perp$: projection on $T\mathbb{S}^n$

So geodesics of $(\mathbb{S}^n, g_{\mathbb{S}^n})$: curves γ such that their acceleration in $\mathbb{R}^{n+1}$ is orthogonal to $T\mathbb{S}^n$ $\Rightarrow$ great circles of $\mathbb{S}^n$



p, q: not antipodal

γ_2: <u>Not</u> length minimizing.

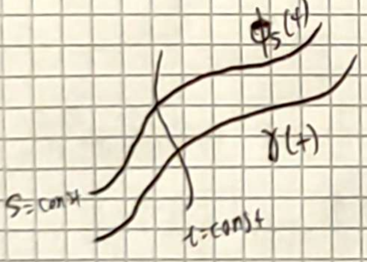
(Recall: On (M, g): $d(p,q) = \inf\{\ell(\gamma) : \gamma(0)=p, \gamma(1)=q\}$.

<u>Definition</u>: A smooth <u>variation</u> of a curve $\gamma : [a,b] \longrightarrow M$ is a smooth map $\phi : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow M$ such that

$$\underset{t}{[a,b]} \times \underset{s}{(-\varepsilon,\varepsilon)}$$

- $\phi(t, 0) = \gamma(t) \quad \forall t \in [a,b]$

<u>Notation</u>: $\phi_s(t) = \phi(t,s)$ (so $\phi_0(t) = \gamma(t)$) also: $\gamma_s(t) = \phi_s(t)$



<u>Define</u>: $\dfrac{\partial \phi}{\partial t} := d\phi\left(\dfrac{\partial}{\partial t}\right)$: tangent vector to the curve $t \longrightarrow \phi_s(t)$

$\dfrac{\partial \phi}{\partial s} := d\phi\left(\dfrac{\partial}{\partial s}\right)$: tangent vector to the curve $s \longrightarrow \phi_s(t)$

(Variation vector field).

- In local coordinates: $\left(\dfrac{\partial \phi}{\partial t}\right)^k = \dfrac{\partial \phi^k}{\partial t}$, $\left(\dfrac{\partial \phi}{\partial s}\right)^k = \dfrac{\partial \phi^k}{\partial s}$

- Strictly speaking: $\dfrac{\partial \phi}{\partial t}, \dfrac{\partial \phi}{\partial s}$ cannot always be extended to vector fields on M

(In analogy with vector fields along a curve)

Lemma: $\nabla_{\frac{\partial\phi}{\partial t}} \frac{\partial\phi}{\partial s} = \nabla_{\frac{\partial\phi}{\partial s}} \frac{\partial\phi}{\partial t}$, where ∇ is the Levi-Civita connection.

Proof: In local coordinates

$$\left(\nabla_{\frac{\partial\phi}{\partial t}} \frac{\partial\phi}{\partial s}\right)^k \underset{\substack{\text{along the curve} \\ t \to \phi_s(t)}}{\overset{\frac{\partial\phi}{\partial t}\text{: derivative}}{=\!=\!=}} \frac{\partial}{\partial t}\left(\frac{\partial\phi^k}{\partial s}\right) + \Gamma^k_{ij} \frac{\partial\phi^i}{\partial t}\frac{\partial\phi^j}{\partial s}$$

Similarly: $$\left(\nabla_{\frac{\partial\phi}{\partial s}} \frac{\partial\phi}{\partial t}\right)^k = \frac{\partial}{\partial s}\left(\frac{\partial\phi^k}{\partial t}\right) + \Gamma^k_{ij} \frac{\partial\phi^i}{\partial s}\frac{\partial\phi^j}{\partial t}$$

since $\frac{\partial}{\partial s}\frac{\partial}{\partial t}\phi^k = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\phi^k$

and $\Gamma^k_{ij} = \Gamma^k_{ji}$ the two expressions are equal.

Alternative proof:

On the manifold $[a,b] \times (-\varepsilon, \varepsilon)$, we have

$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$.

So $0 = d\phi\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]\right) = \left[d\phi\left(\frac{\partial}{\partial t}\right), d\phi\left(\frac{\partial}{\partial s}\right)\right] = \left[\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial s}\right]$.

since ∇ is symmetric. The result follows. ▨

Lemma: If γ is non-singular ($\dot\gamma \neq 0$), the length function $s \to l(\gamma_s)$ is differentiable near $s=0$ and satisfies

$$① \quad \frac{d}{ds} l(\phi_s) = \int_a^b \frac{\left\langle \nabla_{\frac{\partial\phi}{\partial t}} \frac{\partial\phi}{\partial s}, \frac{\partial\phi}{\partial t}\right\rangle}{\left\|\frac{\partial\phi}{\partial t}\right\|} dt$$

Proof

$$l(\gamma_s) = \int_a^b \sqrt{\langle \dot{\gamma}_s, \dot{\gamma}_s \rangle}\, dt = \int_a^b \sqrt{\left\langle \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle}\, dt$$

$$\frac{d}{ds} l(\gamma_s) = \int_a^b \frac{\frac{\partial}{\partial s}\left\langle \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle}{2\sqrt{\left\langle \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle}}\, dt$$

$$= \int_a^b \frac{\left\langle \nabla_{\frac{\partial \phi}{\partial s}} \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle}{\left\| \frac{\partial \phi}{\partial t} \right\|}\, dt$$

$$\overset{\text{Previous lemma}}{=} \int_a^b \frac{\left\langle \nabla_{\frac{\partial \phi}{\partial t}} \frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t} \right\rangle}{\left\| \frac{\partial \phi}{\partial t} \right\|}\, dt \qquad \square$$

Theorem 1st variation of the length

Let $\gamma : [a,b] \longrightarrow M$ be smooth with $\|\dot{\gamma}\| = \text{const}$. For any smooth variation of γ $\phi : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow M$,

$$\frac{d}{ds} l(\phi_s)\Big|_{s=0} = \frac{b-a}{l(\gamma)} \left(\left\langle \frac{\partial \phi}{\partial s}, \dot{\gamma} \right\rangle \Big|_{t=a}^{t=b} - \int_a^b \left\langle \frac{\partial \phi}{\partial s}\Big|_{t=0}, D_{\dot{\gamma}} \dot{\gamma} \right\rangle dt\right.$$

Proof of the theorem:

Since ~~$l(\gamma)$ (and $l(\phi_s)$) is independent of the parametrization. Assume without loss of generality~~

~~that~~ $\|\dot\gamma\| = \text{const}$, $\Rightarrow$ $l(\gamma) = (b-a)\cdot\|\dot\gamma\|$.

From (1):

$$\frac{d}{ds} l(\phi_s)\Big|_{s=0} = \int_a^b \frac{\langle \nabla_t \frac{\partial\phi}{\partial s}\big|_{s=0}, \dot\gamma\rangle}{\|\dot\gamma\|}\, dt = \frac{1}{\|\dot\gamma\|}\int_a^b \langle \nabla_{\dot\gamma} \frac{\partial\phi}{\partial s}\big|_{s=0}, \dot\gamma\rangle\, dt$$

$$= \frac{1}{\|\dot\gamma\|}\int_a^b \left(\frac{d}{dt}\langle \frac{\partial\phi}{\partial s}\big|_{s=0}, \dot\gamma\rangle - \langle \frac{\partial\phi}{\partial s}\big|_{s=0}, \nabla_{\dot\gamma}\dot\gamma\rangle\right) dt$$

$$= \underbrace{\frac{1}{\|\dot\gamma\|}}_{\frac{b-a}{l(\gamma)}}\left(\langle \frac{\partial\phi}{\partial s}\big|_{s=0}, \dot\gamma\rangle\Big|_{t=a}^{b} - \int_a^b \langle \frac{\partial\phi}{\partial s}\big|_{s=0}, \nabla_{\dot\gamma}\dot\gamma\rangle\, dt\right).$$

$\square$

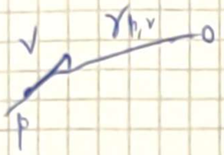
Remark:

Geodesic: $\nabla_{\dot\gamma}\dot\gamma = 0 \Rightarrow \|\dot\gamma\| = \text{const}$

If I reparametrize: $\gamma' = \gamma \circ h$: $\underbrace{\nabla_{\dot\gamma'}\dot\gamma' \parallel \dot\gamma'}_{\text{Equation for } \cancel{\text{geod}} \text{ unparametrized geodesic.}}$

Recall: $\forall p \in M,\ v \in T_pM$: $\exists!$ maximal geodesic

$\gamma_{p,v}$ with $\gamma_{p,v}(0) = p,\ \dot{\gamma}_{p,v}(0) = v$



$\Omega_p \subseteq T_pM$: The set of $v \in T_pM$ s.t. $\gamma_{p,v}$ exists on $[0,1]$

Rmk: - $0 \in \Omega_p$: $\gamma_{p,0}(t) = \{p\}$ $\forall t$

- If $v \in \Omega_p$, $\lambda \in [0,1]$: $\lambda v \in \Omega_p$

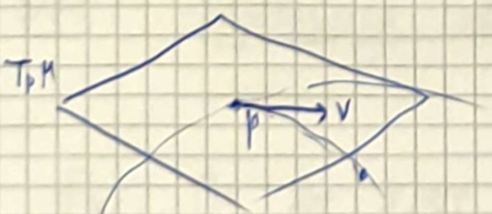
since $\gamma_{p,\lambda v}(t) = \gamma_{p,v}(\lambda t)$ ← $\gamma_{p,v}(0) = p$, $\frac{d}{dt}(\gamma_{p,v}(\lambda t))\big|_{t=0} = \lambda \cdot v$

If $\gamma_{p,v}$ it is defined for $t \in [0,1]$, $\gamma_{p,\lambda v}$ is defined for $t \in [0, \frac{1}{\lambda}] \supseteq [0,1]$

So: Ω_p is <u>star-shaped</u>

<u>Def</u>: $\exp_p : \Omega_p \longrightarrow M$

$\exp_p(v) = \gamma_{p,v}(1)$



So: $\boxed{\exp_p(tv) = \gamma_{p,v}(t)}$

$\exp_p$: Maps straight lines through 0 to geodesics through p

<u>Lemma</u>: $\exp_p$ is a local diffeomorphism around $v = 0$.

Proof: By the inverse function theorem, it suffices to show that

$$d\exp_p\big|_{v=0} \text{ is invertible}$$

$$d\exp_p\big|_{v=0} : \underbrace{T_{v=0}T_pM}_{\substack{\cong \\ T_pM}} \longrightarrow \underset{\substack{\exp_p(0) \\ \| \\ p}}{T_pM}$$

We will show that $d\exp_p\big|_{v=0} = id_{T_pM}$

For any map $F: N \to M$

If γ is a curve in N, $dF\big|_{\gamma(0)}(\dot\gamma(0)) = \frac{d}{dt}(F\circ\gamma)\big|_{t=0}$

$$\exp_p : \underbrace{t\xi}_{\gamma(t)} \longrightarrow \gamma_{p,\xi}(t) \qquad \dot\gamma(0)=\xi$$

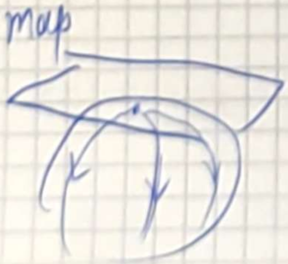
so $$d\exp_p\big|_{v=0}(\underbrace{\dot\gamma(0)}_{\xi}) = \frac{d}{dt}\gamma_{p,\xi}(t)\big|_{t=0} = \xi \qquad \forall\, \xi\in T_pM$$

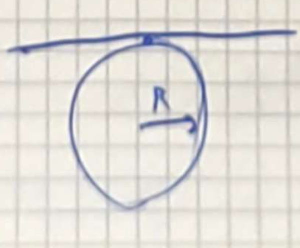
$\square$

Def: Injectivity radius at p: $i(p) = \sup\{r>0 : \exp_p \text{ is a diffeo on } B_r(0)\subseteq T_pM\}$

Ex: On the sphere S^2 with the round metric:

Exponential map





$i(p) = \pi R$

Normal coordinates

Identify T_pM with $\mathbb{R}^n$ via an orthonormal (with respect to $g|_p$) basis $e_1, \dots, e_n$

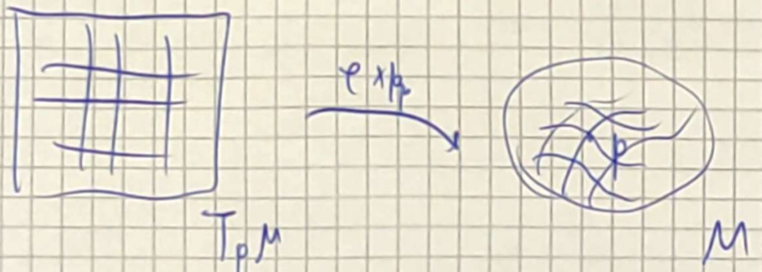
So $w = w^i e_i$ $\forall w \in T_pM$ ← "Cartesian" coordinates on T_pM

Normal coordinates in a neighborhood of p $\Rightarrow$ push-forward of Cartesian coordinates on T_pM via $\exp_p$

i.e. $\forall q$ in a small neighborhood of p: If $q = \exp_p v$

$$\boxed{x^i(q) = v^i}$$

$$x^i(q) = (\exp_p^{-1}(q))^i$$



$\Rightarrow \forall v \in T_pM$ near 0:

$$x^i(\exp_p v) = v^i$$

$$\Rightarrow \underbrace{x^i(\exp_p(tv))}_{\gamma^i_{p,v}(t)} = t \cdot v^i$$

Proposition: In normal coordinates centered at p:

$$g_{ij}(0) = \delta_{ij}, \quad \Gamma^k_{ij}(0) = 0, \quad \partial_k g_{ij}(0) = 0$$

(But in general: $\partial^2 g \neq 0$ $\Rightarrow$ this is related to the curvature tensor)